Analysis of a Generalized Retrial System with Coupled Orbits

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***Abstract*—We study a single-server retrial queueing model with N classes of customers following independent Poisson inputs. A class-i customer, which meets server busy, joins a type-i orbit. Then orbital customers try to occupy the server using a modified constant retrial policy called *coupled orbit queues policy*. Namely, the orbit i retransmits a class-i customer to server after an exponentially distributed time with a rate which depends in general on the binary states (busy or not) of other orbits $j \neq i$. The service times have general class-dependent distribution and the model is described by a non-Markov regenerative process. This model is motivated by increase the impact of wireless interference. We apply regenerative approach and local balance equations to obtain necessary stability conditions and some bounds on the important performance measures of the model. Moreover, we suggest also a sufficient stability condition and verify our results numerically by simulation experiments.**

I. Introduction

We describe and study a multiclass single-server retrial queueing model which is fed by independent Poisson inputs of customers. A class-i customer which meets server busy joins an infinite capacity i-orbit queue, $i = 1, \dots, N$. The orbital customer attempts to capture the server according to a *modified* constant retrial policy called *coupled orbits*, see [9]. More exactly, as described in [9], the orbit i retransmits a customer after an exponentially distributed time with rate μ_i, if *there exists at least one other non-empty orbit*. Otherwise, that is when all other orbits are empty, orbit i switches retransmission rate to μ_i^*, $i = 1, \dots, N$. Note that in a classic constant retrial rate system the retransmission rate remains unchanged and, in particular, does not depend on the state of given (and other) orbits. (A detailed description of the classic constant retrial rate system can be found in [4], [6], while the modified retrial model is described in [9], [16], [15], [17].)

This work extends the analysis developed in [9] to a more general retrial policy. More exactly, the retransmission rate of orbit i in general may depend on the *current configuration of busy and empty (other) orbits*, respectively. To the best of our knowledge, this general model is completely new and it is the main contribution of this research. Furthermore, we prove necessary stability condition and obtain some stationary performance measures and the corresponding bounds. Besides, by analogy with [9], we formulate and verify by simulation a sufficient stability condition. Following [9], we apply the regenerative approach to analyze more general system with coupled orbits.

The retrial queues have been extensively studied in the literature, see the books [13], [3], and the survey papers [2], [18]. The stability analysis of a single-server, multi-class retrial queue with constant retrial rates based on the regenerative approach has been developed in [5], [7]. We mention the recent work [8] on stability analysis of the multiclass system with classic retrial rate policy, where regenerative analysis has been applied as well. The retrial systems with coupled orbits [15], [16], [17], have potential applications in the modelling of wireless multiple access systems, in particular, relay-assisted cognitive cooperative wireless systems [22]. In such a system users transmit packets to a common destination node, and the orbit queues play a role of relay nodes to retransmit blocked packets, see for instance, [22]. In this regard we recall that there is need for developing the cognitive radio communication, to solve a spectrum underutilization problem [20], [22]. In the modern cognitive radio [20] a wireless node is capable to obtain the status operational environment, and it opens a possibility to dynamically adjusts operational parameters (say, retransmission rates) to achieve full spectrum utilization [10], [12], [15].

As another example we mention that in cellular networks, the available transmission rate (in a particular cell) decreases as the number of users in the neighboring cells increase [10]. A similar effect arises in the processor sharing models, see [11], [19].

This paper is organized as follows. In Section 2 we describe our model and present some preliminary results. In Section 3, we develop the regenerative stability analysis of the stationary retrial system with coupled orbits. As a result, we obtain also the necessary stability conditions of this system. An important ingredient of this research is contained in Section IV, where we formulate and discuss sufficient stability condition of this model. In section V, we demonstrate simulation results to support our theoretical results. In particular, we verify by simulation both necessary and sufficient stability conditions for $N = 3$ classes of customers.

II. Description of the model

We consider a bufferless single-server, multiclass retrial queueing model with N classes of primary customers. The class-i customers follow Poisson input with rate λ_i, $i = 1, \ldots, N$. It is assumed that all inputs are independent, and it means that we can consider instead the summary Poisson input with the rate

$$\lambda := \sum_{i=1}^{N} \lambda_i,$$

in which case an arbitrary arrival is class-i customer with the probability $p_i =: \lambda_i/\lambda,\ i = 1, \ldots, N$. Denote $\{t_n\}$ the arrival instants of the merged input (Poisson) stream with rate λ, and introduce interarrival times $\tau_n = t_{n+1} - t_n$, $n \geq 1$. Then it follows that $\{\tau_n\}$ are independent identically distributed (iid) exponential variables with expectation $\mathsf{E}\tau = 1/\lambda \in (0, \infty)$. To denote a generic element of an iid sequence we will omit the corresponding serial index. Also we consider the iid service times of class-i customers, $\{S_n^{(i)}, n \geq 1\}$, with service rate

$$\gamma_i =: \frac{1}{\mathsf{E}S^{(i)}} \in (0,\, \infty),\ i = 1, \ldots, N.$$

The key feature of this model is that the retrial rates are *state-dependent* because the (top) customer from orbit i makes a retrial attempt after an exponentially distributed time with a rate depending on the current status of other orbits: *busy or empty*. More exactly, we consider N-dimensional vectors $J(i) = \{j_1, \ldots, j_N\}$, where the ith component $j_i = 1$ and each component $j_k \neq j_i$ belongs to the binary set $\{0, 1\}$, that is $J(i) \subset \{0, 1\}^N$. It is assumed that if $j_k = 1$, then the k-th orbit is busy, otherwise, if $j_k = 0$, then orbit k is empty. We call $J(i)$ a binary configuration of the orbits, or *configuration* in short. Because, in each configuration $J(i)$, orbit i is always busy, we actually will consider $(N-1)$-dimensional configuration $J(i)$, omitting position $j_i = 1$. Define the set $\mathcal{G}(i) = \{J(i)\}$ of all possible configurations $J(i)$ (in which orbit i is busy). It is assumed that if, at some instant t, the configuration is $J(i)$, then the retransmission rate from orbit i is a *given constant* $\mu_{J(i)}$. In general, different configurations have different retransmission rates. It is convenient to introduce $M_i = \{\mu_{J(i)} : J(i) \in \mathcal{G}(i)\}$, the set of possible rates for all configurations in which orbit i is busy.

Thus, in the current setting, the retransmission rate of a given orbit may depend on any possible configurations and it makes this setting much more general than that in [9]. (Recall the model from [9]: orbit i has rate μ_i if at least one (other) orbit is busy, otherwise, the rate is μ_i^*, $i = 1, \ldots, N$.) In summary, this work demonstrates new possibilities of the analysis method developed in [9] for a less general model.

Now we introduce basic notation. Denote $S(t)$ the remaining service time of a customer at instant t^- (we put $S(t) = 0$, if the server is free). Then

$$I(t) = \int_0^t \mathsf{1}(S(u) = 0)du,$$

is the summary idle time of the server in interval $[0,\, t]$, where $\mathsf{1}$ denotes indicator function. Denote by $A_i(t)$ the number of class-i arrivals in interval of time $[0,\, t]$. Then the summary work which class-i customers bring in the system in interval $[0,\, t]$ is

$$V_i(t) := \sum_{n=1}^{A_i(t)} S_n^{(i)}, \tag{1}$$

and then the summary work arrived in $[0,\, t]$ equals

$$V(t) := \sum_{i=1}^{N} V_i(t) = \sum_{i=1}^{N} \sum_{n=1}^{A_i(t)} S_n^{(i)},\ t \geq 0. \tag{2}$$

Denote by $N_i(t)$ the number of class-i orbital customers and by $W_i(t)$ the remaining work (workload) in orbit i, at instant t^-, $i = 1, \ldots, N$. We will consider the basic (one-dimensional) process $X(t) := N(t) + Q(t)$, $t \geq 0$, where $Q(t) \in \{0,\, 1\}$ is the number of customers in the server at instant t^-, and $N(t) := \sum_i N_i(t)$ is the sum of orbit sizes at instant t^-. In order to study the process $X := \{X(t),\, t \geq 0\}$ we show that it is a regenerative process. Indeed, denote $X(t_n) = X_n$, and let $T_0 := 0$. Now we define the following instants:

$$T_{n+1} = \inf\Big(t_k > T_n : X_k = 0\Big),\ n \geq 0.$$

In other words, at each instant T_n a customer arrives in an empty system. Then it is easy to see that the random elements

$$G_n := \{X(t) : T_n \leq t < T_{n+1}\},\ n \geq 0,$$

are iid and the distribution of G_n is independent of n. It means that the instants $\{T_n\}$ are classical regenerations of the basic process X [23]. Then $\{G_n\}$ are called regeneration cycles of the process X and $T_{n+1} - T_n$ are iid *regeneration periods*. We denote T the generic period. The process X is called *positive recurrent* if the first regeneration period is finite, that is $T_1 < \infty$ with probability 1 (w.p.1), and the mean generic period is finite, that is $\mathsf{E}T < \infty$ [1]. It is worth mentioning that the positive recurrence is the key step to prove the existence the stationary regime of the system by the regenerative method, see [1], [21]. Thus we treat positive recurrence as the stationarity of the basic process (and our retrial system).

III. Analysis of stationary regime

In this section, we consider stationary system and obtain the necessary stability conditions. Recall that stationarity (stability) means that the basic regenerative process X is positive recurrent, $\mathsf{E}T < \infty$. Using the so-called local balance equations and Strong Law of Large Numbers (SLLN) for the renewal processes, we obtain some important performance measures and bounds describing stationary regime of the model. Because the input is Poisson, the interarrival time is exponential (and hence, *non-lattice*), then there exist all limits in distribution we consider below [1]. In particular, $\mathsf{P}(S(t) = 0) \to \mathsf{P}_0 = 1 - \mathsf{P}_b$, where P_b is the stationary *busy probability* of the server and P_0 is the stationary *idle probability*. Denote the traffic intensity for each class,

$$\rho_i = \lambda_i/\gamma_i,\, i = 1, \dots, N.$$

Let $B(t)$ be the busy time of the server in interval $[0,\, t]$, so $B(t) = t - I(t)$.

We obtain the following balance equation connecting, for each instant t, the arrived work, the departed work and the remaining work, respectively,

$$V(t) = \sum_{i=1}^{N} W_i(t) + S(t) + t - I(t),\ t \geq 0, \tag{3}$$

where $W_i(t)$ is the remaining work in the system at instant t related to class-i customers. By the positive recurrence, it follows from theory of regenerative processes (see [23]) that with probability 1 (w.p.1)

$$\sum_{i=1}^{N} W_i(t) + S(t) = o(t),\ \ t \to \infty.$$

By the SLLN, w.p.1, see (1),

$$\lim_{t\to\infty} \frac{V_i(t)}{t} = \lim_{t\to\infty} \frac{\sum_{n=1}^{A_i(t)} S_n^{(i)}}{A_i(t)} = \mathsf{E}S^{(i)},\ i = 1, \dots, N. \tag{4}$$

It immediately gives

$$\lim_{t\to\infty} \sum_{i=1}^{N} \frac{\sum_{n=1}^{A_i(t)} S_n^{(i)}}{A_i(t)} = \sum_{i=1}^{N} \mathsf{E}S^{(i)} = \sum_{i=1}^{N} \frac{1}{\gamma_i}. \tag{5}$$

By the SLLN,

$$\lim_{t\to\infty} \frac{A_i(t)}{t} = \lambda_i,$$

and we obtain, by (2) the following limit expression:

$$\lim_{t\to\infty} \frac{V(t)}{t} = \lim_{t\to\infty} \sum_{i=1}^{N} \frac{\sum_{n=1}^{A_i(t)} S_n^{(i)}}{A_i(t)} \frac{A_i(t)}{t} = \sum_{i=1}^{N} \rho_i. \tag{6}$$

We observe that the busy time $B(t)$ is a *cumulative* process with the positive recurrent process of regenerations $\{T_n\}$. Then the stationary busy probability P_b can be obtained as the (w.p.1) limit [23]:

$$\lim_{t\to\infty} \frac{B(t)}{t} = \mathsf{P}_b. \tag{7}$$

As a result, we obtain from (3), (6), (7) the following important relation

$$\mathsf{P}_b = \sum_{i=1}^{N} \rho_i =: \rho \leq 1. \tag{8}$$

Denote $B_i(t)$ the busy time the server is occupied by class-i customers, in interval $[0,\, t]$. Then it follows from the balance equations

$$V_i(t) = W_i(t) + B_i(t),\, i = 1, \dots, N. \tag{9}$$

Because $W_i(t) = o(t)$, $t \to \infty$, and the limit

$$\lim_{t\to\infty} \frac{B_i(t)}{t} = \mathsf{P}_b^{(i)} \tag{10}$$

is the stationary probability the server is occupied by class-i customer, then we obtain from (9) in the limit that

$$\mathsf{P}_b^{(i)} = \rho_i,\ i = 1, \dots, N. \tag{11}$$

Now we prove that inequality (8) is indeed strict.

Theorem 1. If the multiclass retrial system with coupled orbits is positive recurrent, then

$$\mathsf{P}_b = \rho < 1. \tag{12}$$

Proof. For this setting, the proof from [9] remains unchanged but we outline it for easy reading. Denote by I_0 the duration of an empty period within a regeneration cycle. Let also B be a generic busy period, i.e., the time the server is busy within a regeneration cycle. Then, the regeneration period can be expressed as $T =_{st} B + I_0$. It then easy to show as in [9] that $\mathsf{E}I_0 > 0$. (To prove $\mathsf{E}I_0 > 0$, we denote $\max_i S^{(i)} = \zeta$ and show that $\mathsf{E}I_0 \geq \delta\mathsf{P}(\tau > \zeta + \delta) > 0$ for some $\delta > 0$. For more details see [9].) Denote $I(t) = t - B(t)$ the idle time of server in interval $[0,\, t]$. Then w.p.1,

$$\lim_{t\to\infty} \frac{I(t)}{t} = \frac{\mathsf{E}I_0}{\mathsf{E}T} = \mathsf{P}_0 = 1 - \mathsf{P}_b > 0, \tag{13}$$

and strong inequality (12) follows. □

Introduce the maximal possible rate from orbit i:

$$\hat{\mu}_i = \max_{J(i)\in\mathcal{G}(i)} \mu_{J(i)}.$$

The following statement contains the *necessary stability condition* and is a generalization of the corresponding result obtained in [9] for a less general model.

Theorem 2. Assume that the N-class retrial system with coupled orbits is positive recurrent. Then,

$$\mathsf{P}_b = \sum_{i=1}^{N} \rho_i = \rho \leq \min_{1\leq i\leq N} \left[\frac{\hat{\mu}_i}{\lambda_i + \hat{\mu}_i}\right] < 1. \tag{14}$$

Proof. Denote indicator $\mathsf{1}_{J(i)}(t) = 1$, if, at instant t, the orbits have a (fixed) configuration $J(i)$ and *server is free*, and $\mathsf{1}_{J(i)}(t) = 0$, otherwise. Then

$$T_{J(i)}(t) =: \int_0^t \mathsf{1}_{J(i)}(u)du.$$

is the summary time, in interval $[0,\, t]$ when the system has configuration $J(i)$ and server is idle. Then

$$T_0^{(i)}(t) := \sum_{J(i)\in\mathcal{G}(i)} T_{J(i)}(t)$$

is the summary time, in interval $[0,\, t]$, when *server is idle and orbit i is busy*, allowing successful retrials from orbit i.

To proceed with our analysis, we introduce indicator $I_k^{(i)} = 1$, if the kth class-i customer joins orbit i, and $I_k^{(i)} = 0$, otherwise. Then the number of class-i customers, $A_i^{(0)}(t)$, joining orbit i in interval $[0,\, t]$, equals

$$A_i^{(0)}(t) = \sum_{k=1}^{A_i(t)} I_k^{(i)}.$$

By the SLLN for the renewal process, $A_i(t)/t \to \lambda_i$, and it then follows by positive recurrence that, as $t \to \infty$,

$$\frac{A_i^{(0)}(t)}{t} = \frac{A_i(t)}{t} \frac{1}{A_i(t)} \sum_{k=1}^{A_i(t)} I_k^{(i)} \to \lambda_i \mathsf{P}_b, \qquad (15)$$

where the limit

$$\lim_{t\to\infty} \frac{1}{A_i(t)} \sum_{k=1}^{A_i(t)} I_k^{(i)} = \mathsf{P}_b \qquad (16)$$

exists (by the positive recurrence) and is the stationary busy probability of server. We note that indicators in (15) are *dependent* in general. The r.h.s. of (16) is independent of i by the property PASTA [1]. This property means that the limiting fraction of (Poisson) class-i customers, which see server busy (and join orbit), equals the limiting fraction of busy time of server. Because the latter fraction is independent of class of customers, the limit in (16) is independent of i as well.

Now we establish a balance relation between the number of customers $A_i^{(0)}(t)$ joining orbit i and the number of customers $D_i(t)$ leaving this orbit, in interval $[0,\, t]$.

Denote by $\hat{D}_{J(i)}(t)$ the Poisson process of retrials from orbit i provided, in interval $[0,\, t]$, server is free and $1_{J(i)}(u) = 1$ for all $u \in [0,\, t]$. Note that this process has rate $\mu_{J(i)}$. By the property of Poisson process, we obtain the following *stochastic relations* for $i = 1, \ldots, N$:

$$\begin{aligned} A_i^{(0)}(t) &= N_i(t) + D_i(t) \\ &= {}_{st} N_i(t) + \sum_{J(i)\in\mathcal{G}(i)} \hat{D}_i(T_{J(i)}(t)). \end{aligned} \qquad (17)$$

Note that $T_{J(i)}(t) \to \infty$ as $t \to \infty$, $i = 1, \ldots, N$. By the renewal theory, for each configuration $J(i)$,

$$\frac{\hat{D}_i(T_{J(i)}(t))}{T_{J(i)}(t)} \to \mu_{J(i)},\ t \to \infty. \qquad (18)$$

Notice that the limit

$$\lim_{t\to\infty} \frac{T_{J(i)}(t)}{t} =: \mathsf{P}_0^{J(i)} \qquad (19)$$

also exists and is the stationary probability that the system has configuration $J(i)$ (so orbit i is busy) and server is free. An important observation is that

$$\sum_{J(i)\in\mathcal{G}(i)} \mathsf{P}_0^{J(i)} = \lim_{t\to\infty} \frac{T_0^{(i)}(t)}{t} =: \mathsf{P}_0^{(i)} \qquad (20)$$

is the stationary probability that *server is free and orbit i is busy*. By the positive recurrence, $N_i(t) = o(t)$, $t \to \infty$, and, by (17)-(19), we obtain the following equations

$$\lambda_i \mathsf{P}_b = \sum_{J(i)\in\mathcal{G}(i)} \mathsf{P}_0^{J(i)} \mu_{J(i)},\ i = 1, \ldots, N. \qquad (21)$$

It is easy to verify that each orbit can be occupied with a positive probability while server remains free, implying strict inequality $\mathsf{P}_0^{(i)} < \mathsf{P}_0$. Now we obtain from (21)

$$\begin{aligned} \lambda_i \mathsf{P}_b &\le \hat{\mu}_i \sum_{J(i)\in\mathcal{G}(i)} \mathsf{P}_0^{J(i)} \\ &= \hat{\mu}_i \mathsf{P}_0^{(i)} < \hat{\mu}_i \mathsf{P}_0 \\ &= \hat{\mu}_i (1 - \mathsf{P}_b),\ i = 1, \ldots, N. \end{aligned} \qquad (22)$$

To obtain the upper bound as tight as possible, it remains to take $\min$ over i in (22), to obtain (14). □

Because of (20) we obtain from (21) the following two-sided inequalities which, unlike (12), *are not uniform in i*.

Theorem 3. The following inequalities hold

$$\frac{\lambda_i}{\hat{\mu}_i}\rho \le \mathsf{P}_0^{(i)} \le \frac{\lambda_i}{\mu_i^0}\rho,\ i = 1, \ldots, N. \qquad (23)$$

As in the paper [9], relation (12) can be extended to the corresponding m-server retrial system with equivalent servers. In this case, the r.h.s. in inequality (12) (and in (8)) is replaced by m, because the term $t - I(t)$ on r.h.s. of (3) is replaced by $mt - \sum_{k=1}^m I_k(t)$, where $I_k(t)$ is the summary idle time of server k in interval $[0,\, t]$. We formulate it as the following statement.

Theorem 4. The necessary stability condition of m-server N-class retrial system with coupled orbits is

$$\sum_{i=1}^N \rho_i < m. \qquad (24)$$

Moreover, the stationary busy probability $\mathcal{P}_b$ of an *arbitrary server* in this case equals

$$\mathcal{P}_b = \frac{\sum_{i=1}^N \rho_i}{m}. \qquad (25)$$

IV. Sufficient stability condition

Now we formulate and discuss sufficient stability conditions, and compare them with the necessary ones. In the next section we verify by simulation the accuracy of these conditions.

Denote, for each class i, the minimal retrial rate

$$\mu_i^0 = \min_{J(i)\in\mathcal{G}(i)} \mu_{J(i)}.$$

Following [9], we formulate the following *sufficient stability condition* of the single-server retrial system with N coupled orbits:

$$\sum_{i=1}^N \rho_i + \max_{1\le i\le N} \frac{\lambda}{\mu_i^0 + \lambda} < 1. \qquad (26)$$

As in [9], this form of condition is motivated by the requirement to have negative drift of the workload process. The full proof of condition (26) will be presented in a separate work. We can rewrite (26) as

$$\sum_i \rho_i < 1 - \max_{1\le i\le N} \frac{\lambda}{\mu_i^0+\lambda} = \min_i \left(\frac{\mu_i^0}{\lambda+\mu_i^0}\right),$$

and thus the sufficient stability condition becomes

$$\rho = \sum_i \rho_i < \min_i \left(\frac{\mu_i^0}{\lambda+\mu_i^0}\right). \tag{27}$$

Now we compare (27) with the necessary stability condition (14). Observe that the function $f(x) = x/(\lambda+x)$ is monotonically increases in x and $\max_i \lambda_i < \lambda$. Then, because $\mu_i^0 \le \hat{\mu}_i$, we obtain

$$\min_i \frac{\mu_i^0}{\lambda+\mu_i^0} < \min_i \frac{\mu_i^0}{\lambda_i+\mu_i^0} \le \min_i \frac{\hat{\mu}_i}{\lambda_i+\hat{\mu}_i}. \tag{28}$$

These bounds allow us to calculate the *gap* between the necessary and sufficient conditions,

$$\Delta =: \min_i \frac{\hat{\mu}_i}{\lambda_i+\hat{\mu}_i} - \min_i \frac{\mu_i^0}{\lambda+\mu_i^0} > 0. \tag{29}$$

We note that dependence of Δ on parameters of the model has been studied in [9] for two-orbit system.

It has been proved in [7], using a dominating loss system with the *saturated orbits*, that (14) is stability criterion for the symmetric model (when $\mu_0^{(i)} \equiv \mu$ and $\lambda_i \equiv \lambda/N$) with a special class of service time distributions. At the same time, in the symmetric case (31) becomes

$$\Delta = \frac{\mu}{\lambda/N+\mu} - \frac{\mu}{\lambda+\mu},$$

and $\Delta = 0$ if and only if $N = 1$. It shows that we can not achieve stability criterion by the approach given above.

Remark 1. Consider the *non-coupled* orbits, when $\mu_{J(i)} = \mu_i$ for all configurations $J(i)$. In other words, it is a conventional retrial system in which each busy orbit i has a fixed retrial rate μ_i. Then relations (21) and (23) become

$$\lambda_i \mathsf{P}_b = \mu_i \mathsf{P}_0^{(i)},$$

and we obtain the following explicit formula for the stationary probability that orbit queue i is busy and server is idle:

$$\mathsf{P}_0^{(i)} = \frac{\lambda_i}{\mu_i}\sum_{k=1}^{N} \rho_k, \;\; i = 1, \dots, N. \tag{30}$$

In this setting the difference Δ becomes

$$\Delta = \min_i \frac{\mu_i}{\lambda_i+\mu_i} - \min_i \frac{\mu_i}{\lambda+\mu_i}. \tag{31}$$

V. Simulation results

In this section we verify by simulation some obtained above theoretical results for a 3-class system (with three orbits). We will focus on the verification of the stability conditions. To this end, we define the following variables,

$$\begin{aligned} \Gamma_1 :=& \;\; \min_i \tfrac{\hat{\mu}_i}{\lambda_i+\hat{\mu}_i} - \rho, \\ \Gamma_2 :=& \;\; \min_i \tfrac{\mu_i^0}{\lambda+\mu_i^0} - \rho. \end{aligned} \tag{32}$$

which describe the proximity of the state of the system to the boundary of stability region. In particular, $\Gamma_i > 0,\ i = 1, 2$, means that both stability conditions (14) and (27) are satisfied, and a stable dynamics of the obits are expected in this case. Conversely, if $\Gamma_i < 0,\ i = 1, 2$, then we expect instability of the orbits. Below these observations are supported by simulation.

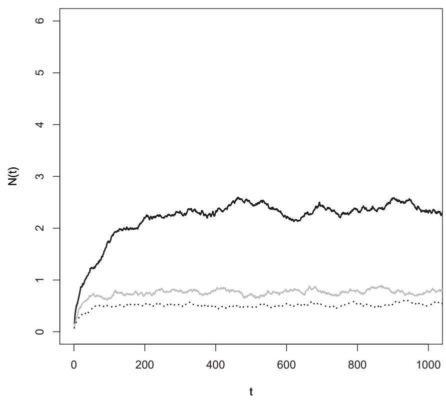

Fig. 1. Exponential service time. Conditions (14) and (27) hold: $\Gamma_1 > 0$, $\Gamma_2 > 0$; all orbits are stable

Now we present numerical results obtained by simulation, to illustrate stability/instability of the orbits depending on whether both necessary and sufficient conditions, (14), (27), are met, or not. This is verified for exponential and Pareto service time distribution. In all figures, the black, grey and (grey) dotted curve corresponds to the 1st, 2nd and 3rd orbit, respectively.

We show the dynamics of orbits as time t increases. (More exactly, the axis t shows the used number of discrete "events" in the applied discrete-event simulation algorithm: arrivals, departures, attempts.)

Recall that $i_k = 1\,(0)$ means that orbit k is busy (empty). Moreover it is assumed that, in each configuration $J(i)$, the remaining two orbits are considered in an increasing order. Because we consider three orbits, then the capacity of each set $\mathcal{M}_i$ and set $\mathcal{G}(i)\}$ equals 4. Indeed, for the two remaining orbits $j < k$ with $k, j \neq i$ the following four configurations $J(i)$ are possible:

$$\mathcal{G}(i) = \Big\{(i_j = 0,\, i_k = 0),\, (i_j = 1,\, i_k = 0), \\ (i_j = 0,\, i_k = 1),\, (i_j = 1,\, i_k = 1)\Big\}. \quad (33)$$

In the *1st experiment*, see Fig. 1, we use the following values of the input and service rates:

$$\lambda_1 = 5,\ \lambda_2 = 3,\ \lambda_3 = 2, \quad (34)$$

$$\gamma_1 = 30,\ \gamma_2 = 10,\ \gamma_3 = 20, \quad (35)$$

and in *all experiments* we use the following retrial rates:

$$M_1 = \Big\{\mu^1_{00} = 20,\, \mu^1_{10} = 20,\, \mu^1_{01} = 25,\, \mu^1_{11} = 25\Big\},$$
$$M_2 = \Big\{\mu^2_{00} = 30,\, \mu^2_{10} = 40,\, \mu^2_{01} = 35,\, \mu^2_{11} = 42\Big\},$$
$$M_3 = \Big\{\mu^3_{00} = 10,\, \mu^3_{10} = 15,\, \mu^3_{01} = 20,\, \mu^3_{11} = 25\Big\}, \quad (36)$$

where, in notation μ^i, the status of concrete configuration is reflected. For instance, $\mu^1_{01} = 25$ is the retrial rate of configuration $J(1) = (i_2 = 0,\, i_3 = 1)$, while $\mu^3_{10} = 15$ is the retrial rate of configuration $J(3) = (i_1 = 1,\, i_2 = 0)$.

One can calculate that in the 1st experiment, $\rho = 0.56$ and, as a result, $\Gamma_1 = 0.27, \Gamma_2 = 0.1$. Therefore, both necessary and sufficient conditions, (14), (27), are satisfied, and as we see on Fig. 1, all three orbits are stable, as expected.

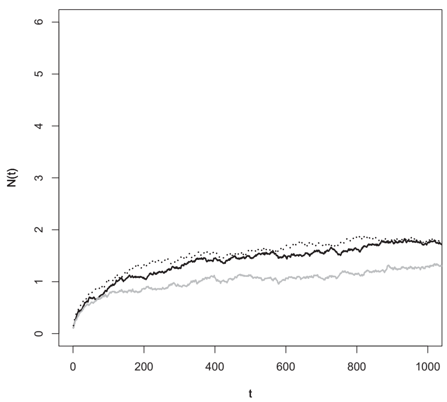

Fig. 2. Pareto service time. Conditions (14) and (27) hold: $\Gamma_1 > 0$, $\Gamma_2 > 0$; all orbits are stable

Fig.2 shows the dynamics of orbits when service time $S^{(i)}$ of class-i customer has *Pareto distribution*:

$$F_i(x) = 1 - (\frac{x^i_0}{x})^\alpha,\ x \geq x^i_0\ \ (F_i(x) = 0,\ x \leq x^i_0),\ x^i_0 > 0,$$

with expectation

$$\mathsf{E}S^{(i)} = \frac{\alpha\, x^i_0}{\alpha - 1},\ \alpha > 1,\ i = 1, 2, 3. \quad (37)$$

We select $\alpha = 2$ and the following values of the shape parameter x^i_0 for orbit $i = 1, 2, 3$, respectively:

$$x^i_0 = \frac{1}{60},\ \frac{1}{20},\ \frac{1}{40}. \quad (38)$$

Because $\gamma = 1/\mathsf{E}S$, then this choice gives the same service rates as in the 1st experiment (see (35)):

$$\gamma_1 = 30,\ \gamma_2 = 10,\ \gamma_3 = 20. \quad (39)$$

In this case $\rho = 0.56$ and moreover, $\Gamma_1 = 0.27, \Gamma_2 = 0.1$. It again implies stability of all orbits, see Fig.2.

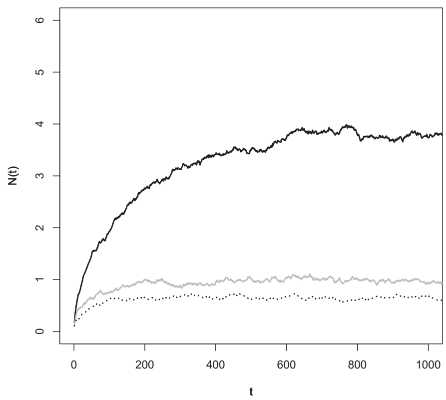

Fig. 3. Exponential service time. Condition (14) holds, $\Gamma_1 > 0$; condition (27) is violated, $\Gamma_2 < 0$; all orbits are stable

Fig. 3 shows the dynamics of orbits for the input rates

$$\lambda_1 = 5,\ \lambda_2 = 3,\ \lambda_3 = 2,$$

and exponential service rates

$$\gamma_1 = 20,\ \gamma_2 = 10,\ \gamma_3 = 20.$$

In this case,

$$\rho = 0.68,\ \ \Gamma_1 = 0.18,\ \ \Gamma_2 = -0.02, \quad (40)$$

that is condition (14) holds, while condition (27) is *violated.* As we see, all orbits are still stable, however the stability is reached at a higher level (at least for orbit 1). Thus, simulation confirms that i) the sufficient stability condition for our three-orbit system remains the same as condition found for two-orbit system in [9]; and that ii) a violation of the sufficient condition not always leads to instability. (This is an expected result because sufficient and necessary conditions are different.)

Fig. 4 shows the dynamics of the orbits in the exponential model with the rates

$$\lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 2,$$
$$\gamma_1 = 10, \gamma_2 = 10, \gamma_3 = 20.$$

Then $\rho = 0.9$ and $\Gamma_1 = -0.07, \Gamma_2 = -0.17$, that is both condition (14) and (27) are violated. It is interesting that in

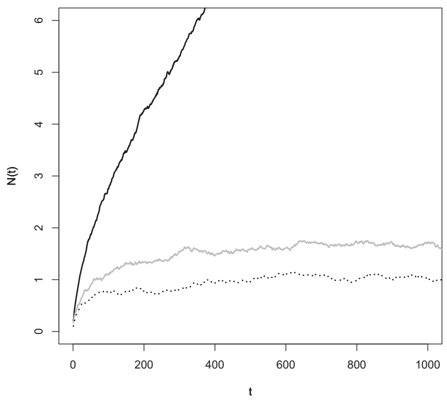

Fig. 4. Exponential service times. $\Gamma_1 < 0$, $\Gamma_2 < 0$, conditions (14) and (27) are violated; the 1st orbit is unstable

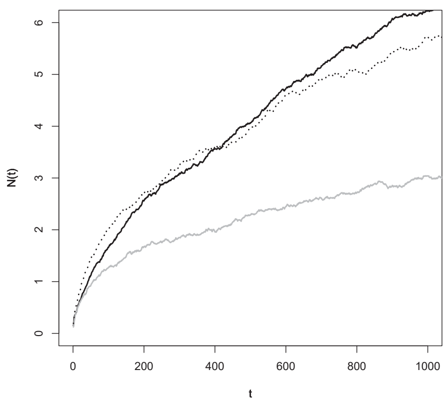

Fig. 5. Exponential service time. Both conditions (14) and (27) are violated: $\Gamma_1 < 0$, $\Gamma_2 < 0$; all orbits are unstable

this case only the 1st orbit is unstable. We suggest that it is because the 1st orbit has the maximal traffic intensity ρ_1.

Fig.5 illustrates the behavior of orbits in the exponential model with the rates

$$\lambda_1 = 3, \lambda_2 = 3, \lambda_3 = 4;$$
$$\gamma_1 = 10, \gamma_2 = 7, \gamma_3 = 20.$$

Then $\rho = 0.92$, implying $\Gamma_1 = -0.09, \Gamma_2 = -0.26$, that is again both condition (14) and (27) are violated. However, unlike Fig. 4, *all three orbits become unstable* in this experiment. We suggest that it is because the chosen parameters make the model closer to the symmetric model.

Fig. 6 demonstrates the correctness of theoretical result (27) for the *non-coupled orbits and Pareto service time* with $\alpha = 2$ and shape parameters (38). Then service rates are (see (39)):

$$\gamma_1 = 30,\ \gamma_2 = 10,\ \gamma_3 = 20,$$

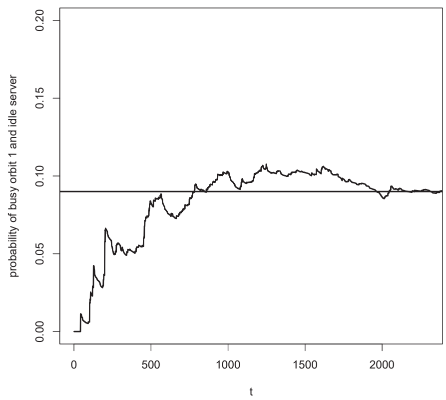

Fig. 6. Pareto service time. Estimation the probability $\mathsf{P}_0^{(1)} = \mathsf{P}$(busy orbit 1, idle server) for *non-coupled orbits*

while the input rates are taken as in previous experiment:

$$\lambda_1 = 3,\ \lambda_2 = 3,\ \lambda_3 = 4.$$

Moreover, the following retrial rates are used:

$$\mu_1 = 20,\ \mu_2 = 30,\ \mu_3 = 10.$$

One can check that in this case $\rho = 0.6$, $\Gamma_1 = 0.11$, $\Gamma_2 = -0.1$. This is very close to parameters in experiment 3, see (40), and the system must be stable (although condition (27) *is violated*). Indeed, as Fig 6 shows, the model remains stable since the sample mean estimate converges to theoretical value of the stationary probability

$$\mathsf{P}_0^{(1)} = \mathsf{P}(\text{busy orbit 1, idle server}) = 0.09,$$

given by equality (30). We notice that this experiment confirms the convergence of the estimates of the corresponding probabilities for two other orbits as well. (Because of similarity, we omit the corresponding numerical illustration.)

In summary, it is worth mentioning that our experiments confirm stability conditions (14), (27) for the multi-obit system with *non-exponential service time*.

VI. Conclusion

We consider a multi-class (multi-orbit) bufferless retrial system with N classes of retrial customers following independent Poisson inputs and the so-called coupled orbits. A class-i customer meeting busy server joins an infinite capacity orbit $i = 1, 2, ..., N$. The head customer in orbit i attempts to occupy server after an exponentially distributed time with rate $\mu_{J(i)}$ where $J(i) = (j_1, \dots, j_N)$ is a fixed configuration of the binary states of all other orbits, $j_k \in \{0, 1\}$ (empty or busy), provided that $j_i = 1$. It means that, in each configuration $J(i)$, orbit i is always busy. This setting is a considerable generalization of that has been studied in [9], where only the following possibilities have been considered: i) all orbits $j \neq i$ are idle, ii) at least one orbit $j \neq i$ is not idle. We apply regenerative approach to derive necessary stability condition. Also sufficient stability condition is formulated. We verify by simulation some stationary performance measures and the accuracy of the found stability conditions.

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REFERENCES

[1] Asmussen, S.: Applied Probability and Queues. Wiley, N.Y. (1987)

[2] Artalejo, J.R.: Accessible bibliography on retrial queues: Progress in 2000-2009. Mathematical and Computer Modelling **51**(9-10), 1071-1081 (2010)

[3] Artalejo, J.R., Gomez-Corral, A.: Retrial Queueing Systems: A Computational Approach. Springer (2008)

[4] Avrachenkov, K., Yechiali, U.: Retrial networks with finite buffers and their application to internet data traffic. Prob. Engin. Infor. Sc. **22**, 519-536 (2008)

[5] Avrachenkov, K., Morozov, E., Nekrasova, R., Steyaert, B.: Stability analysis and simulation of N-class retrial system with constant retrial rates and Poisson inputs. Asia Pac. J. Oper. Res. **31**(2), 1440002 (18 pages) (2014)

[6] Avrachenkov, K., Nain, P., Yechiali, U.: A retrial system with two input streams and two orbit queues. Queueing Syst. **77**(1), 1-31 (2014)

[7] Avrachenkov, K., Morozov, E., Steyaert, B.: Sufficient stability conditions for multi-class constant retrial rate systems. Queueing Syst. **82**, 149-171 (2016)

[21] Morozov, E., Delgado, R.: Stability analysis of regenerative queues. Automation and Remote control **70**, 1977-1991 (2009)

[8] Morozov, E., Phung-Duc, T.: Stability analysis of a multiclass retrial system with classical retrial policy. Perform. Eval. doi:10.1016/j.peva.2017.03.003 (2017)

[9] Morozov, E., Dimitriou, I. Stability Analysis of a Multiclass Retrial System with Coupled Orbit Queues, Proceedings of 14th European Workshop, EPEW 2017, Berlin, Germany, September 7-8, 2017, $DOI10.1007/978-3-319-66583-2_6$

[10] Bonald, T., Borst, S., Hegde, N., Proutiere, A.: Wireless data performance in multi-cell scenarios, in: Proc. ACM Sigmetrics/Performance '04, pp. 378-388, ACM, New York (2004)

[11] Bonald, T., Massoulié, L., Proutiére, A., Virtamo, J.: A queueing analysis of max-min fairness, proportional fairness and balanced fairness. Queueing Syst. **53**(1-2), 65-84 (2006)

[12] Borst, S., Jonckheere, M., Leskela, L.: Stability of parallel queueing systems with coupled service rates, Discrete Event Dyn. S. **18**(4), 447-472 (2008)

[13] Falin G.I., Templeton, J.G.D.: Retrial Queues. Chapman & Hall, London (1997)

[14] Feller, W.: An Introduction to Probability Theory, vol. 2. Wiley, N.Y (1971)

[15] Dimitriou, I.: A two class retrial system with coupled orbit queues. Prob. Engin. Infor. Sc. **31**(2), 139-179 (2017)

[16] Dimitriou, I.: A queueing system for modeling cooperative wireless networks with coupled relay nodes and synchronized packet arrivals. Perform. Eval. doi: 10.1016/j.peva.2017.04.002 (2017)

[17] Dimitriou, I.: Modeling and analysis of a relay-assisted cooperative cognitive network. In: N. Thomas, M. Farshaw (eds.), ASMTA 2017, LNCS, Springer, Newcastle, UK (2017)

[18] Kim, J., Kim, B.: A survey of retrial queueing systems. Annals of Operations Research **247**(1), 3-36 (2016)

[19] Liu, X., Chong, E., Shroff, N.: A framework for opportunistic scheduling in wireless networks. Comp. Netw. **41**, 451-474 (2003)

[20] Mitola, J., Maguire, G.: Cognitive radio: making software radios more personal. IEEE Pers. Commun. **6**(4), 13-18 (1999)

[22] Sadek, A., Liu, K., Ephremides, A.: Cognitive multiple access via cooperation: Protocol design and performance analysis. IEEE Trans. Infor. Th. **53**(10), 3677-3696 (2007)

[23] Smith, W.L.: Regenerative stochastic processes. Proceedings of Royal Society, Ser. A **232**, 6-31 (1955)