# Algebraic Solutions to Scheduling Problems in Project Management

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**Abstract**

We offer a computational approach to schedule development based on models and methods of idempotent algebra. The approach allows one to represent different types of precedence relationships among activities as linear vector equations written in terms of an idempotent semiring. As a result, many issues in project scheduling can be reduced to solving computational problems in the idempotent algebra setting, including linear equations and the eigenvalue-eigenvector problem. We give solutions to the problems in a compact vector form that provides a basis for the development of efficient computation algorithms and related software applications, including those intended for parallel implementation.

## 1 Introduction

The problem of scheduling a large-scale set of activities is a key issue in project management [1, 2]. There is a variety of project-scheduling techniques developed to deal with different aspects of the problem, ranging from the classical Critical Path Method and the Program Evaluation and Review Technique marked the beginning of the active research in the area in 1950s, to more recent methods of idempotent algebra [4, 5, 6, 7, 10].

We offer a new computational approach [10] to schedule development based on implementation of models and methods of idempotent algebra. The approach allows one to represent different types of precedence relationships among activities as linear vector equations written in terms of an idempotent semiring. As a result, many issues in project scheduling can be reduced to solving computational problems in the idempotent algebra setting, including linear equations and the eigenvalue-eigenvector problem. We give solutions to the problems in a compact vector form that provides a basis for the development of efficient computation algorithms and related software applications, including those intended for parallel implementation.

The rest of the paper is organized as follows. We start with algebraic definitions and notation, and then outline basic results that underlie subsequent applications of idempotent algebra. Furthermore, examples of actual problems in project scheduling are considered. We show how to formulate the problems in an algebraic setting, and give related algebraic solutions. Finally, concluding remarks are given to summarize the results.

# 2 Definitions and Notation

We start with a brief overview of basic concepts, terms and symbols in idempotent algebra. Further details can be found in [4, 5, 6, 7, 10].

## 2.1 Idempotent Semiring

Consider a set $\mathbb{X}$ that is equipped with two operations $\oplus$ and $\otimes$ referred to as addition and multiplication, and has neutral elements $\mathbb{0}$ and $\mathbb{1}$ called zero and identity. We suppose that $\langle\mathbb{X},\mathbb{0},\mathbb{1},\oplus,\otimes\rangle$ is a commutative semiring, where addition is idempotent and multiplication is invertible. Such a semiring is usually called idempotent semifield.

Let us define $\mathbb{X}_{+}=\mathbb{X}\setminus\{\mathbb{0}\}$. Each $x\in\mathbb{X}_{+}$ is assumed to have its inverse $x^{-1}$. For any $x\in\mathbb{X}_{+}$ and integer $p>0$, the power is defined in the ordinary way,

$$x^{0}=\mathbb{1},\quad x^{p}=x^{p-1}\otimes x=x\otimes x^{p-1},\quad x^{-p}=(x^{-1})^{p},\quad \mathbb{0}^{p}=\mathbb{0}.$$

In what follows, the multiplication sign $\otimes$ is omitted as is usual in conventional algebra. The notation of power is thought of as defined in terms of idempotent algebra. However, in the expressions that represent exponents, we use ordinary arithmetic operations.

Since the addition is idempotent, it induces a partial order $\leq$ on $\mathbb{X}$ according to the rule: $x\leq y$ if and only if $x\oplus y=y$. The relation symbols are understood below in the sense of this partial order. According to the order, it holds that has $x\geq\mathbb{0}$ for any $x\in\mathbb{X}$.

As a classical example of idempotent semirings (semifields), one can consider the semiring

$$\mathbb{R}_{\max,+}=\langle\mathbb{R}\cup\{-\infty\},-\infty,0,\max,+\rangle.$$

The semiring has the neutral elements $\mathbb{0}=-\infty$ and $\mathbb{1}=0$. For each $x\in\mathbb{R}$, there exists its inverse $x^{-1}$, which is equal to $-x$ in ordinary arithmetics.

For any $x, y \in \mathbb{R}$, the power $x^y$ is equivalent to the arithmetic product $xy$. The order induced by idempotent addition coincides with the natural linear order on $\mathbb{R}$.

The semiring $\mathbb{R}_{\max,+}$ provides the basis for the development of algebraic solutions to project scheduling problems in subsequent sections.

## 2.2 Matrix Algebra

Now consider matrices with elements in $\mathbb{X}$. The set of all matrices of size $m \times n$ is denoted by $\mathbb{X}^{m\times n}$.

The matrix with all entries equal to zero is the null matrix denoted by $\mathbb{0}$. A matrix is called regular if it has at least one nonzero element in every row.

For any scalar $x \in \mathbb{X}$ and matrices

$$A = (a_{ij}) \in \mathbb{X}^{m\times n}, \qquad B = (b_{ij}) \in \mathbb{X}^{m\times n}, \qquad C = (c_{ij}) \in \mathbb{X}^{n\times l}$$

matrix addition and multiplication, as well as multiplication by scalars are defined in the usual way with the expressions

$$\{A \oplus B\}_{ij} = a_{ij} \oplus b_{ij}, \qquad \{BC\}_{ij} = \bigoplus_{k=1}^{n} b_{ik}c_{kj}, \qquad \{xA\}_{ij} = xa_{ij}.$$

A square matrix is called diagonal if all its off-diagonal entries are zero, and triangular if its entries above (below) the diagonal are zero. The matrix $I = \mathrm{diag}(\mathbb{1}, \ldots, \mathbb{1})$ is referred to as identity matrix.

A matrix $A$ is irreducible if and only if it cannot be put in a block triangular form by simultaneous permutations of rows and columns.

For any square matrix $A$ and integer $p > 0$, the power is defined as usual,

$$A^0 = I, \qquad A^p = A^{p-1}A = AA^{p-1}.$$

For a square matrix $A = (a_{ij}) \in \mathbb{X}^{n\times n}$, its trace is given by

$$\mathrm{tr}\, A = \bigoplus_{i=1}^{n} a_{ii}.$$

Let $A = (a_{ij}) \in \mathbb{X}^{m\times n}$ be a regular matrix. The pseudo-inverse matrix of $A$ is defined as $A^{-} = (a_{ij}^{-}) \in \mathbb{X}^{n\times m}$, where $a_{ij}^{-} = a_{ji}$ if $a_{ji} \neq \mathbb{0}$, and $a_{ij} = \mathbb{0}$ otherwise.

Finally, consider the set $\mathbb{X}^n$ of all column vectors with elements in $\mathbb{X}$. The vector with all elements equal to zero is called null vector and denoted by $\mathbb{0}$.

For any column vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T \neq \mathbb{0}$, one can define a row vector $\boldsymbol{x}^- = (x_1^-, \ldots, x_n^-)$ with elements $x_i^- = x_i$ if $x_i \neq \mathbb{0}$, and $x_i = \mathbb{0}$ otherwise, $i = 1, \ldots, n$.

We define the distance between any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{X}_+^n$ with a metric

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}^- \boldsymbol{x} \oplus \boldsymbol{x}^- \boldsymbol{y}.$$

When $\boldsymbol{y} = \boldsymbol{x}$ we have the minimum distance $\rho(\boldsymbol{x}, \boldsymbol{x}) = \mathbb{1}$.

In the semiring $\mathbb{R}_{\max,+}^n$, the metric takes the form

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = \max_i |x_i - y_i|,$$

and thus coincides with the Chebyshev metric.

# 3 Preliminary Results

Now we outline some basic results from [8, 9, 10] that underlie subsequent applications of idempotent algebra to project scheduling.

## 3.1 The Equation $A\boldsymbol{x} = \boldsymbol{d}$

Suppose a matrix $A \in \mathbb{X}^{m \times n}$ and a vector $\boldsymbol{d} \in \mathbb{X}^m$ are given. Let $\boldsymbol{x} \in \mathbb{X}^n$ be an unknown vector. We examine the equation

$$A\boldsymbol{x} = \boldsymbol{d}, \tag{3.1}$$

and the inequality

$$A\boldsymbol{x} \leq \boldsymbol{d}, \tag{3.2}$$

A solution $\boldsymbol{x}_0$ to equation (3.1) or inequality (3.2) is called the maximum solution if $\boldsymbol{x}_0 \geq \boldsymbol{x}$ for all solutions $\boldsymbol{x}$ of (3.1) or (3.2).

We present a solution to (3.1) based on analysis of distance between vectors in $\boldsymbol{X}^n$. To simplify further formulae, we use the notation

$$\Delta = (A(\boldsymbol{d}^- A)^-)^- \boldsymbol{d}.$$

**Lemma 1.** *Suppose $A \in \mathbb{X}^{m \times n}$ is a regular matrix, and $\boldsymbol{d} \in \mathbb{X}_+^m$ is a vector without zero components. Then it holds that*

$$\min_{\boldsymbol{x} \in \mathbb{X}_+^n} \rho(A\boldsymbol{x}, \boldsymbol{d}) = \Delta^{1/2},$$

*where the minimum is achieved at $\boldsymbol{x}_0 = \Delta^{1/2} (\boldsymbol{d}^- A)^-$.*

**Lemma 2.** *Under the same conditions as in Lemma 1 it holds that*

$$\min_{A\boldsymbol{x}\leq\boldsymbol{d}} \rho(A\boldsymbol{x},\boldsymbol{d}) = \min_{A\boldsymbol{x}\geq\boldsymbol{d}} \rho(A\boldsymbol{x},\boldsymbol{d}) = \Delta,$$

*where the first minimum is achieved at* $\boldsymbol{x}_1 = (\boldsymbol{d}^{-}A)^{-}$, *and the second at* $\boldsymbol{x}_2 = \Delta(\boldsymbol{d}^{-}A)^{-}$.

As a consequence of Lemma 1 and 2, we get the following result.

**Theorem 1.** *A solution of equation* (3.1) *exists if and only if* $\Delta = \mathbb{1}$. *If solvable, the equation has the maximum solution given by*

$$\boldsymbol{x} = (\boldsymbol{d}^{-}A)^{-}.$$

Suppose that $\Delta > \mathbb{1}$. In this case equation (3.1) has no solution. However, we can define a pseudo-solution to (3.1) as a solution of the equation

$$A\boldsymbol{x} = \Delta^{1/2}A(\boldsymbol{d}^{-}A)^{-},$$

which is always exists and takes the form $\boldsymbol{x}_0 = \Delta^{1/2}(\boldsymbol{d}^{-}A)^{-}$. It follows from Lemma 1 that the pseudo-solution yields the minimum deviation between vectors $\boldsymbol{y} = A\boldsymbol{x}$ and the vector $\boldsymbol{d}$ in the sense of the metric $\rho$.

Consider the problem of finding two vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ that provide the minimum deviation between both sides of (3.1), while satisfying the respective inequalities $A\boldsymbol{x} \leq \boldsymbol{d}$ and $A\boldsymbol{x} \geq \boldsymbol{d}$. As it is easy to see from Lemma 2, these vectors are given by

$$\boldsymbol{x}_1 = (\boldsymbol{d}^{-}A)^{-}, \qquad \boldsymbol{x}_2 = \Delta(\boldsymbol{d}^{-}A)^{-}.$$

The next statement is another consequence of the above results.

**Lemma 3.** *For any regular matrix* $A$ *and vector* $\boldsymbol{d}$ *without zero components, the solution to* (3.2) *is given by the inequality*

$$\boldsymbol{x} \leq (\boldsymbol{d}^{-}A)^{-}.$$

A solution to equation (3.1) with an arbitrary matrix $A$ and a vector $d$ is considered in [10]

### 3.2 The Equation $A\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$

Suppose a matrix $A \in \mathbb{X}^{n\times n}$ and a vector $\boldsymbol{b} \in \mathbb{X}^n$ are given, and $\boldsymbol{x} \in \mathbb{X}^n$ is an unknown vector. Consider the equation

$$A\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}, \tag{3.3}$$

and the inequality

$$A\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}, \tag{3.4}$$

A solution to equation (3.3) is proposed based on application of a function $\operatorname{Tr} A$ that takes each square matrix to a scalar and plays the role of the determinant in conventional linear algebra. The function is given by

$$\operatorname{Tr} A = \bigoplus_{m=1}^{n} \operatorname{tr} A^m$$

and exploited to examine whether the equation has a unique solution, many solutions, or no solution.

For any $A \in \mathbb{X}^{n\times n}$, we define matrices $A^+$ and $A^\times$ as follows

$$A^+ = I \oplus A \oplus \cdots \oplus A^{n-1}, \qquad A^\times = AA^+ = A \oplus \cdots \oplus A^n.$$

Let $\boldsymbol{a}_i^+$ be column $i$ of $A^+$, and $a_{ii}^\times$ be entry $(i,i)$ of $A^\times$. Provided that $\operatorname{Tr} A = \mathbb{1}$, we define the matrix $A^* = (\boldsymbol{a}_i^*)$ with the columns

$$\boldsymbol{a}_i^* = \begin{cases} \boldsymbol{a}_i^+, & \text{if } a_{ii}^\times = \mathbb{1}, \\ \mathbb{0}, & \text{otherwise.} \end{cases}$$

If $\operatorname{Tr} A \neq \mathbb{1}$, then we take $A^* = \mathbb{0}$.

The solution to equation (3.3) is given by the following result.

**Theorem 2.** *Let $\boldsymbol{x}$ be the solution of equation* (3.3) *with an irreducible matrix $A$. Then the following statements hold:*

1) *if $\operatorname{Tr} A < \mathbb{1}$, then there exists a unique solution $\boldsymbol{x} = A^+\boldsymbol{b}$;*

2) *if $\operatorname{Tr} A = \mathbb{1}$, then $\boldsymbol{x} = A^+\boldsymbol{b} \oplus A^*\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{X}^n$;*

3) *if $\operatorname{Tr} A > \mathbb{1}$, then with the condition $\boldsymbol{b} = \mathbb{0}$, there exists only the solution $\boldsymbol{x} = \mathbb{0}$, whereas with $\boldsymbol{b} \neq \mathbb{0}$, there is no solution.*

**Lemma 4.** *Let $\boldsymbol{x}$ be the solution of inequality* (3.4) *with an irreducible matrix $A$. Then the following statements hold:*

1) *if $\operatorname{Tr} A \leq \mathbb{1}$, then $\boldsymbol{x} = A^+(\boldsymbol{b} \oplus \boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{X}^n$;*

2) *if $\operatorname{Tr} A > \mathbb{1}$, then with the condition $\boldsymbol{b} = \mathbb{0}$, there exists only the solution $\boldsymbol{x} = \mathbb{0}$, whereas with $\boldsymbol{b} \neq \mathbb{0}$, there is no solution.*

Related results for the case of arbitrary matrices can be found in [8, 10].

### 3.3 Eigenvalues and Eigenvectors

A scalar $\lambda$ is an eigenvector of a square matrix $A \in \mathbb{X}^{n\times n}$ if there is a vector $\boldsymbol{x} \in \mathbb{X}^n \setminus \{\mathbb{0}\}$ such that

$$A\boldsymbol{x} = \lambda\boldsymbol{x}.$$

The maximum eigenvalue is called spectral radius of $A$ and given by

$$\varrho = \bigoplus_{m=1}^{n} \operatorname{tr}^{1/m}(A^m).$$

The eigenvector corresponding to $\varrho$ takes the form

$$\boldsymbol{x} = A_\varrho^* \boldsymbol{v},$$

where $A_\varrho = \varrho^{-1}A$, and $\boldsymbol{v}$ is any vector.

**Lemma 5.** *For any irreducible matrix $A$ with the spectral radius $\varrho$, it holds that*

$$\min_{\boldsymbol{x}\in\mathbb{X}_+^n} \rho(A\boldsymbol{x}, \boldsymbol{x}) = \varrho \oplus \varrho^{-1},$$

*where the minimum is achieved at any eigenvector $\boldsymbol{x}$ corresponding to $\varrho$.*

The case of arbitrary matrices is considered in [9, 10].

## 4 Applications to Project Scheduling

In this section we show how to apply the algebraic results presented above to solve scheduling problems under various constraints (for further details on the schedule development in project management see, e.g., [1, 2]).

As the underlying idempotent semiring, we use $\mathbb{R}_{\max,+}$ in all examples under discussion.

### 4.1 Precedence Relations of the Start-to-Finish Type

Consider a project that involves $n$ activities. Activity dependencies are assumed the form of Start-to-Finish relations that do not allow an activity to complete until some time after initiation of other activities. The scheduling problem of interest is to find initiation time for all activities subject to given constraints on their completion time.

For each activity $i = 1, \ldots, n$, denote by $x_i$ its initiation time, and by $y_i$ its completion time. Let $d_i$ be a due date, and $a_{ij}$ a minimum possible time lag between initiation of activity $j = 1, \ldots, n$ and completion of $i$.

Given $a_{ij}$ and $d_i$, the completion time of activity $i$ must satisfy the relations

$$y_i = d_i, \qquad x_j + a_{ij} \leq y_i, \quad j = 1, \ldots, n,$$

where if $a_{ij}$ is not actually given for some $j$, it is assumed to be $\mathbb{0} = -\infty$.

The relations can be combined into one equation in the initiation times

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}) = d_i.$$

By replacing the ordinary operations with those in $\mathbb{R}_{\max,+}$ in all equations, we get

$$a_{i1} \otimes x_1 \oplus \cdots \oplus a_{in} \otimes x_n = d_i, \quad i = 1, \ldots, n.$$

For simplicity, we drop the multiplication symbol $\otimes$, and write

$$a_{i1} x_1 \oplus \cdots \oplus a_{in} x_n = d_i, \quad i = 1, \ldots, n.$$

With the notation

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}, \qquad \boldsymbol{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

the scheduling problem under the start-to-finish constraints leads us to solution of the equation

$$A\boldsymbol{x} = \boldsymbol{d}.$$

Consider $\Delta = (A(\boldsymbol{d}^{-}A)^{-})^{-}\boldsymbol{d}$. According to Theorem 1, provided that the condition $\Delta = \mathbb{1} = 0$ is satisfied, the equation has a unique solution $\boldsymbol{x} = (\boldsymbol{d}^{-}A)^{-}$.

If it appears that $\Delta > 0$, then one can compute approximate solutions to the equation

$$\boldsymbol{x}_0 = \Delta^{1/2}(\boldsymbol{d}^{-}A)^{-}, \qquad \boldsymbol{x}_1 = (\boldsymbol{d}^{-}A)^{-}, \qquad \boldsymbol{x}_2 = \Delta(\boldsymbol{d}^{-}A)^{-}.$$

The completion times corresponding to these solution are given by

$$\boldsymbol{y}_0 = A\boldsymbol{x}_0, \qquad \boldsymbol{y}_1 = A\boldsymbol{x}_1 \leq \boldsymbol{d}, \qquad \boldsymbol{y}_2 = A\boldsymbol{x}_2 \geq \boldsymbol{d},$$

and have their deviation from the due dates bounded with

$$\rho(\boldsymbol{y}_0, \boldsymbol{d}) = \Delta^{1/2}, \qquad \rho(\boldsymbol{y}_1, \boldsymbol{d}) = \rho(\boldsymbol{y}_2, \boldsymbol{d}) = \Delta.$$

Suppose that the due date constraints may be adjusted to some extent. As a new vector of due dates, it is natural to take a vector $\boldsymbol{d}'$ such that

$\boldsymbol{y}_1 \leq \boldsymbol{d}' \leq \boldsymbol{y}_2$. In this case, deviation of the new due dates from the original ones does not exceed $\Delta$. The minimum deviation which is equal to $\Delta^{1/2}$ is achieved at $\boldsymbol{d}' = \boldsymbol{y}_0$.

As an example, consider a project with a constraint matrix and two due date vectors given by

$$A = \begin{pmatrix} 8 & 10 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 5 & 4 & 8 \\ 6 & 12 & 11 & 7 \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & 12 \end{pmatrix}, \qquad \boldsymbol{d}_1 = \begin{pmatrix} 14 \\ 11 \\ 16 \\ 15 \end{pmatrix}, \qquad \boldsymbol{d}_2 = \begin{pmatrix} 15 \\ 15 \\ 15 \\ 15 \end{pmatrix}.$$

Fig. 1 demonstrates a network representation of the project.

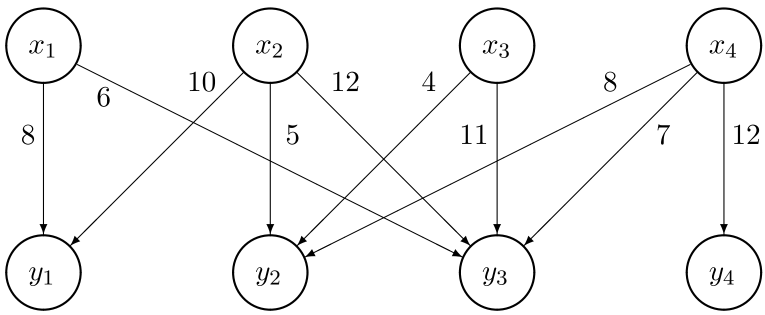


Figure 1: An activity network with Start-to-Finish precedence relations

First we examine the equation $A\boldsymbol{x} = \boldsymbol{d}_1$. Simple calculation gives $\Delta_1 = (A(\boldsymbol{d}_1^- A)^-)^- \boldsymbol{d}_1 = 0$. Therefore, the equation has a unique solution

$$\boldsymbol{x} = (\boldsymbol{d}_1^- A)^- = (6, 4, 5, 3)^T.$$

Consider the equation $A\boldsymbol{x} = \boldsymbol{d}_2$. Since $\Delta_2 = (A(\boldsymbol{d}_2^- A)^-)^- \boldsymbol{d}_2 = 4 > 0$, the equation does not have a solution. Evaluation of approximate solutions gives

$$\begin{aligned} \boldsymbol{x}_0 &= \Delta_2^{1/2}(\boldsymbol{d}_2^- A)^- = (9, 5, 6, 5)^T, & \boldsymbol{y}_0 &= A\boldsymbol{x}_0 = (17, 13, 17, 17)^T, \\ \boldsymbol{x}_1 &= (\boldsymbol{d}_2^- A)^- = (7, 3, 4, 3)^T, & \boldsymbol{y}_1 &= A\boldsymbol{x}_1 = (15, 11, 15, 15)^T, \\ \boldsymbol{x}_2 &= \Delta_2(\boldsymbol{d}_2^- A)^- = (11, 7, 8, 7)^T, & \boldsymbol{y}_2 &= A\boldsymbol{x}_2 = (19, 15, 19, 19)^T. \end{aligned}$$

## 4.2 Precedence Relations of the Start-to-Start Type

Suppose there is a project consisting of $n$ activities and operating under Start-to-Start precedence constraints that determine the minimum (maximum) allowed time intervals between initiation of activities.

For each activity $i = 1, \dots, n$, let $b_i$ be an early possible initiation time, and let $a_{ij}$ be a minimum possible time lag between initiation of activity $j = 1, \dots, n$ and initiation of $i$. The problem is to find the earliest initiation time $x_i$ for every activity $i$ so as to provide for the relations

$$b_i \leq x_i, \qquad a_{ij} + x_j \leq x_i, \quad j = 1, \dots, n,$$

which can be replaced with one equation

$$\max(\max(x_1 + a_{i1}, \dots, x_n + a_{in}), b_i) = x_i.$$

Representation in terms of $\mathbb{R}_{\max,+}$, gives the scalar equations

$$a_{i1}x_1 \oplus \cdots \oplus a_{in}x_n \oplus b_i = x_i, \quad i = 1, \dots, n.$$

With the notation $A = (a_{ij})$, $\boldsymbol{b} = (b_1, \dots, b_n)^T$, $\boldsymbol{x} = (x_1, \dots, x_n)^T$ we arrive at a problem that is to solve the equation

$$A\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}.$$

For simplicity, assume the matrix $A$ to be irreducible. It follows from Theorem 2 that if $\operatorname{Tr} A \leq \mathbb{1} = 0$ then the equation has a nontrivial solution given by $\boldsymbol{x} = A^{+}\boldsymbol{b} \oplus A^{*}\boldsymbol{v}$ for any vector $\boldsymbol{v}$.

Consider a project with start-to-start relations and examine two cases, with and without early initiation time constraints imposed. Let us define a matrix and two vectors as follows

$$A = \begin{pmatrix} 0 & -2 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 0 & 3 & -1 \\ -1 & \mathbb{0} & 0 & -4 \\ 2 & \mathbb{0} & \mathbb{0} & 0 \end{pmatrix}, \qquad \boldsymbol{b}_1 = \mathbb{0}, \qquad \boldsymbol{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}.$$

A graph representation of the project is depicted in Fig. 2.

Let us first calculate the initiation time of activities in the project when $\boldsymbol{b} = \boldsymbol{b}_1 = \mathbb{0}$ (that is, without early initiation time constraints given). Under this assumption, the equation takes the form $A\boldsymbol{x} = \boldsymbol{x}$.

As it is easy to see, the matrix $A$ is irreducible and $\operatorname{Tr} A = 0$. Therefore, the equation has a solution.

Simple algebra gives

$$A^{+} = A^{\times} = \begin{pmatrix} 0 & -2 & 1 & -3 \\ 2 & 0 & 3 & -1 \\ -1 & -3 & 0 & -4 \\ 2 & 0 & 3 & 0 \end{pmatrix}, \qquad A^{*} = \begin{pmatrix} -2 & -3 \\ 0 & -1 \\ -3 & -4 \\ 0 & 0 \end{pmatrix}.$$

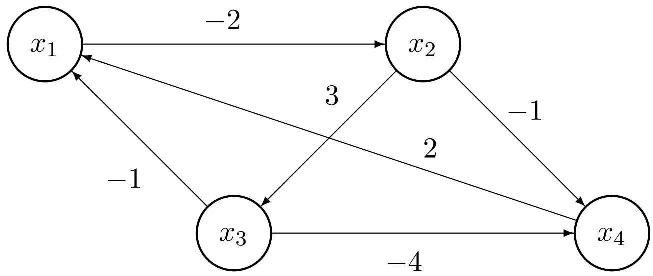


Figure 2: An activity network with Start-to-Start precedence relations

Note that, since $A^+$ and $A^\times$ coincide, one should define $A^* = A^+$. However, considering that the first three columns are proportional, we take only one of them.

The solution to the equation is given by

$$\boldsymbol{x} = A^*\boldsymbol{v} = \begin{pmatrix} -2 & -3 \\ 0 & -1 \\ -3 & -4 \\ 0 & 0 \end{pmatrix}\boldsymbol{v}, \quad \boldsymbol{v} \in \mathbb{R}^2_{\max,+}.$$

Consider the case with the vector $\boldsymbol{b}_2$ and the equation taking the form $A\boldsymbol{x} \oplus \boldsymbol{b}_2 = \boldsymbol{x}$. Now we have

$$A^+\boldsymbol{b}_2 = \begin{pmatrix} 3 \\ 5 \\ 2 \\ 5 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} 3 \\ 5 \\ 2 \\ 5 \end{pmatrix} \oplus \begin{pmatrix} -2 & -3 \\ 0 & -1 \\ -3 & -4 \\ 0 & 0 \end{pmatrix}\boldsymbol{v}, \quad \boldsymbol{v} \in \mathbb{R}^2_{\max,+}.$$

## 4.3 Mixed Precedence Relations

Consider a project that has both Start-to-Finish and Start-to-Start constraints. Let $A_1$ be a given Start-to-Finish constraint matrix, $\boldsymbol{d}$ a vector of due dates, and $\boldsymbol{x}$ an unknown vector of activity initiation time. To meet the constraints, the vector $\boldsymbol{x}$ must satisfy the inequality

$$A_1\boldsymbol{x} \le \boldsymbol{d}.$$

Furthermore, there are also Start-to-Start constraints defined by a constraint matrix $A_2$. This leads to the equation in $\boldsymbol{x}$

$$A_2\boldsymbol{x} = \boldsymbol{x}.$$

Suppose that the equation has a solution $\boldsymbol{x} = A_2^* \boldsymbol{v}$. Substitution of the solution into the above inequality gives

$$A_1 A_2^* \boldsymbol{v} \leq \boldsymbol{d}.$$

Since the maximum solution to the last inequality is $\boldsymbol{v} = (\boldsymbol{d}^- A_1 A_2^*)^-$, the solution to the whole problem is written in the form

$$\boldsymbol{x} = A_2^* (\boldsymbol{d}^- A_1 A_2^*)^-.$$

As an illustration, we evaluate the solution to the problem under the conditions

$$A_1 = \begin{pmatrix} 8 & 10 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 5 & 4 & 8 \\ 6 & 12 & 11 & 7 \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & 12 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & -2 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 0 & 3 & -1 \\ -1 & \mathbb{0} & 0 & -4 \\ 2 & \mathbb{0} & \mathbb{0} & 0 \end{pmatrix},$$

and

$$\boldsymbol{d} = (13, 11, 15, 15)^T.$$

By using results of previous examples, we successively get

$$A_1 A_2^* = \begin{pmatrix} 10 & 9 \\ 8 & 8 \\ 12 & 11 \\ 12 & 12 \end{pmatrix}, \qquad (\boldsymbol{d}^- A_1 A_2^*)^- = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Finally, we have

$$\boldsymbol{x} = A_2^* (\boldsymbol{d}^- A_1 A_2^*)^- = (1, 3, 0, 3)^T.$$

## 4.4 Minimization of the Maximum Flow Time

Assume that a project operates under Start-to-Finish constraints. For each activity in the project, consider the time interval between its initiation and completion, which is usually referred to as the flow time and also as turnaround time or processing time.

In practice, one can be interested in constructing a schedule that minimizes the maximum flow time over all activities in the project. With $\boldsymbol{x}$ standing for a vector of initiation time, and $A$ for a constraint matrix, we arrive at a problem formulated in terms of $\mathbb{R}_{\max,+}$ to find

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \rho(A\boldsymbol{x}, \boldsymbol{x}).$$

It follows from Lemma 5 that the above minimum is equal to the $\varrho \oplus \varrho^{-1}$, where $\varrho$ is the spectral radius of $A$, and it is achieved at the vector given by $\boldsymbol{x} = A_{\varrho}^{*}\boldsymbol{v}$ for any vector $\boldsymbol{v}$.

Suppose a vector $\boldsymbol{d}$ is given to represent activity due dates. Consider a problem of evaluating the latest initiation time for all activities so as to provide both the due date constraints and the minimum flow time condition.

By combining the due date constraints represented in the form

$$A\boldsymbol{x} \leq \boldsymbol{d}$$

with the solution of the minimization problem, we have the inequality

$$AA_{\rho}^{*}\boldsymbol{v} \leq \boldsymbol{d}.$$

With the maximum solution to the inequality $\boldsymbol{v} = (\boldsymbol{d}^{-}AA_{\rho}^{*})^{-}$, we get the solution of the whole problem

$$\boldsymbol{x} = A_{\rho}^{*}(\boldsymbol{d}^{-}AA_{\rho}^{*})^{-}.$$

Let us evaluate the solution with the constraint matrix and due date vector defined as

$$A = \begin{pmatrix} 2 & 4 & 4 \\ 2 & 3 & 5 \\ 3 & 2 & 3 \end{pmatrix}, \qquad \boldsymbol{d} = \begin{pmatrix} 9 \\ 8 \\ 9 \end{pmatrix}.$$

First we get $\rho = 4$, and define the matrix

$$A_{\rho} = \begin{pmatrix} -2 & 0 & 0 \\ -2 & -1 & 1 \\ -1 & -2 & -1 \end{pmatrix}.$$

Furthermore, we have the matrices

$$A_{\rho}^{+} = A_{\rho}^{*} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \qquad A_{\rho}^{*} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Finally, we arrive at the solution

$$\boldsymbol{x} = A_{\rho}^{*}(\boldsymbol{d}^{-}AA_{\rho}^{*})^{-} = \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}.$$

## 5 Conclusion

We have presented an approach that exploits idempotent algebra to solve computational problems in project scheduling. It is shown how to reformulate the problems in an algebraic setting, and then find related solutions based on appropriate results in the idempotent algebra theory. The solutions are given in a compact vector form that provides a basis for the development of efficient computation algorithms and software applications, including those intended for parallel implementation.

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## Bibliography

[1] *A Guide to the Project Management Body of Knowledge: PMBOK Guide.* Newtown Square, PA: Project Management Institute, 2008. 459 p.

[2] K. Neumann, C. Schwindt, J. Zimmermann, *Project Scheduling with Time Windows and Scarce Resources.* Berlin, Springer, 2003. 385 p.

[3] R. Cuninghame-Green, *Minimax Algebra.* Berlin, Springer, 1979. 258 p. (Lecture Notes in Economics and Mathematical Systems, vol. 166)

[4] F. Baccelli, G. Cohen, G. J. Olsder, J.-P. Quadrat, *Synchronization and Linearity: An Algebra for Discrete Event Systems.* Chichester, Wiley, 1993. 514 p.

[5] V. N. Kolokoltsov, V. P. Maslov, *Idempotent Analysis and Its Applications.* N. Y., Springer, 1997. 324 p.

[6] J. S. Golan, *Semirings and Affine Equations Over Them: Theory and Applications.* N. Y., Springer, 2003. 256 p.

[7] B. Heidergott, G. J. Olsder, J. van der Woude, *Max-Plus at Work: Modeling and Analysis of Synchronized Systems.* Princeton, Princeton Univ. Press, 2005. 226 p.

[8] N. K. Krivulin, *Solution of Generalized Linear Vector Equations in Idempotent Algebra* Vestnik St. Petersburg Univ. Math., 39 (1), 2006. pp. 16–26.

[9] N. K. Krivulin, *Eigenvalues and Eigenvectors of Matrices in Idempotent Algebra* Vestnik St. Petersburg Univ. Math., 39 (2), 2006. pp. 72–83.

[10] N. K. Krivulin, *Idempotent Algebra Methods for Problems in Modeling and Analysis of Complex Systems.* St. Petersburg, St. Petersburg Univ. Press, 2009. 256 p. (in Russian)